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Unsteady mass transport from a sphere immersed in a porous medium at finite Peclet numbers

Z.-G. Feng, E. E. Michaelides*

Department of Mechanical Engineering, Tulane University, New Orleans, LA 70018, U.S.A.

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Abstract

A singular perturbation method is employed in order to develop an analytical solution to the problem of the unsteady mass transfer from a sphere immersed in an unbounded saturated porous medium. At the inception of the process, the sphere is suddenly leaking a contaminant, which spreads in the porous medium by convection and diffusion. The boundary conditions at the surface of the sphere are either constant concentration or constant mass flux. Throughout the process the Peclet number is small but finite. The time and length domains of the problem are separated in four sub-domains, which result from the combinations of short and long times, and inner and outer regions. Based on the physical analysis of the problem, the governing equations in these regions are derived and solved in the time domain or the Laplace domain. A matching technique is used to derive the final expressions for the contaminant concentration field and the mass transfer coefficients. Hence, analytical asymptotic solutions for the concentration of the contaminant in the entire space and time domains are derived in terms of the Peclet numbers. The solutions are validated by comparison with known analytical results. © 1998 Elsevier Science Ltd. All rights reserved.

Nomenclature

a radius of the sphere
A_n coefficients defined in (39)
B_n coefficients defined in (43)
c, C concentration functions
D_e effective diffusivity
$\mathbf{e}_r, \mathbf{e}_z, \mathbf{e}_\theta$ unit vectors
erfc complementary error function
E_n functions defined in (48)
F gauge functions
g gravity
$G_n(r, t)$ functions defined in (56)
$H(t)$ Heavyside function
$H_n(S)$ functions defined in (28)
i_n modified Bessel functions
I_n Bessel functions
K permeability
L_I Oseen distance
$L_n(r, T)$ functions defined in (47)
n outward vector
p pressure
P_n Legendre polynomial
Pe Peclet number
r, R radial coordinates
s, S variables in Laplace domain corresponding to t and T
Sh Sherwood number
t, T time variables
U fluid velocity far from the sphere
$\mathbf{v}$ fluid velocity near the sphere
$W(T)$ functions defined in (37)
x, y, z coordinates
$Z_n(r, T)$ functions defined in (45a).

Greek symbols

ζ $\cos\theta$
θ angular coordinate
Λ function defined in (56)
μ fluid viscosity
ρ fluid density
τ_D diffusion time scale
Ψ stream function.

* Corresponding author. Tel.: 001 504 865 5764; Fax: 001 504 862 8747; E-mail: emichael.mailhost@tcs.tulane.edu

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1. Introduction

The subject of heat and mass transfer through porous media is of importance in chemical and environmental processes as well as environmental remediation operations. Heat transfer through rocks or soil, leakage from a vessel with porous insulation around it, contaminant leakage from buried drums, or contaminant leakage from storage underground cavities and their consequent transport through geological strata, are among the several physical processes, where knowledge of the unsteady transport of a scalar quantity (mass of a pollutant) is of importance for remediation. The subject of transport through porous media has been dominated by numerical studies. Among the recent publications, one may mention the study by Nguyen and Paik [1] who obtained numerical results for the unsteady convection from a sphere with variable surface temperature. Pop and Ingham [2] performed a computational study on the natural convection from a sphere in a saturated porous medium, and later Kimura and Pop [3] studied numerically the conjugate convection from a sphere in a porous medium. Recently Yan et al. [4] performed a numerical study of the transient free convection from a sphere enclosed in a porous medium.

There are very few analytical studies on this subject. Among the earlier analytical studies on the steady-state forced heat convection in porous media is Bejan's [5] for the transient temperature field from a point heat source, buried in a fluid-saturated porous medium. Bejan used an asymptotic expansion in terms of *Ra* and derived expressions for the transient (of the order of Ra^1) and steady-state (of the order of Ra^3) heat transfer. A good review on the subject can be found in a more recent monograph by Nield and Bejan [6]. Among the more recent analytical expressions on the subject, Sano and Okihara [7] examined the natural convection around a sphere immersed in a porous medium at very small Rayleigh numbers.

Analytical solutions to any physical process strengthen our understanding and comprehension of the physical mechanisms that play an important role in the processes. They also yield information and insight on the processes, which numerical solutions do not. Analytical solutions are used for the validation of complex numerical codes, which may subsequently be extended and used for the solution of more complex transport problems. In addition, they may yield boundary conditions for certain numerical codes. For these reasons an attempt is made here to derive an analytical solution for the unsteady mass transport problem in a porous medium. A singular perturbation method is used in order to analyze the contaminant transport process in a porous medium, which is due to the sudden leakage from a spherical enclosure. The time and space domain is divided in four sub-domains and asymptotic solutions for the concentration function are obtained in every sub-domain. The matching conditions at the boundaries of the sub-domains are used, in order to derive a generally valid global solution for the contaminant concentration in the porous medium.

The solution technique that will be followed for the problem of mass transfer does not take into account gravity/buoyancy effects. For this reason, the solution developed may not be appropriate for the analogous problem of heat transfer,where natural convection may play an important role in the case of low Peclet number. However,this solution for the mass transfer problem will be applicable to the analogous problem of heat convection from a sphere inside a porous medium, whenever the natural convection may be neglected and there is a correspondence in the governing equations and boundary conditions of the two problems.

2. Basic equations and general method of solution

The main assumptions of this study are as follows:

(a) the sphere, of radius a, is immersed in an unbounded fluid-saturated porous medium;
(b) the velocity field inside the porous medium is governed by Darcy's law and is unidirectional far from the sphere;
(c) the Peclet number throughout the process is finite but small;
(d) buoyancy effects in the fluid are neglected.

The second assumption implies that there may be slip at the surface of the sphere. However, it will be shown in Appendix B that, for the processes considered here, the thickness of the fluid boundary layer at the surface of the sphere is very small in comparison to the radius of the sphere (less than one millionth of the radius of the sphere) and, hence, the effects of this boundary layer on the velocity field may be neglected.

A schematic of the process and of the coordinate system is depicted in Fig. 1. The presence of the sphere creates a disturbance to this velocity field, which is essentially confined to the vicinity of the sphere. Far from the sphere, the velocity field is unidirectional. At the inception of the process ($t = 0$), a contaminant fluid starts leaking from the sphere. The two fluids mix freely and are transported in the porous medium. Without any loss of generality, we consider that the initial concentration of this contaminant in the porous medium is zero (initially 'contaminant-free' medium). After the inception of the leakage process ($t > 0$) we contemplate two boundary conditions at the surface of the sphere: (a) the concentration of the contaminant is constant, c_{s0} and (b) the mass flux is constant. The solution to be derived in the subsequent sections of this manuscript pertains to the constant surface concentration problem. The results

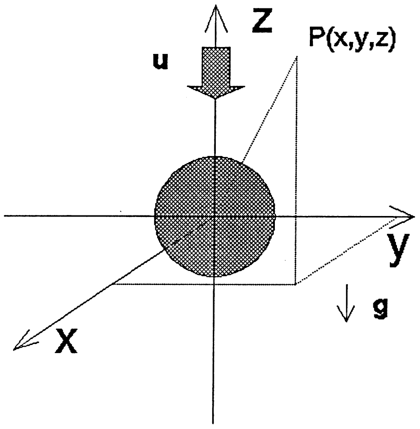

Fig. 1. The flow and the coordinate system.

for the constant mass flux problem are derived by the same method and are simply presented in the last section.

The velocity field $\mathbf{v}$ and the continuity condition of the external fluid, which initially saturates the porous medium, are given by the following equations:

$$\mathbf{v} = -\frac{K}{\mu}\nabla(p+\rho g z), \quad \nabla\cdot\mathbf{v} = 0. \tag{1}$$

The boundary conditions for the external fluid are:

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{at } r = a$$
$$\mathbf{v} = -U\mathbf{e}_z \quad \text{at } r\to\infty. \tag{2}$$

Hence, we obtain the following velocity field for the external fluid (see Appendix A):

$$\mathbf{v} = -U\left(1-\frac{a^3}{r^3}\right)\mathbf{e}_r\mathbf{e}_r\cdot\mathbf{e}_z - U\left(1+\frac{a^3}{2r^3}\right)\mathbf{e}_\theta\mathbf{e}_\theta\cdot\mathbf{e}_z. \tag{3}$$

After the inception of the leakage, at $t = 0$, an unsteady mass transfer process from the sphere commences. The transient diffusion equation governs the distribution of the contaminant in the porous medium. For convenience, we introduce the following dimensionless variables, which are denoted by an asterisk (*). We consider as the characteristic length of the process the radius of the sphere, a, and as characteristic time the diffusion time scale, $\tau_D = a^2/D_c$:

$$(x^*,y^*,z^*) = \left(\frac{x}{a},\frac{y}{a},\frac{z}{a}\right), \quad r^* = \frac{r}{a}, \quad t^* = \frac{tD_e}{a^2},$$
$$c^*(\mathbf{x}^*,t^*) = \frac{c(\mathbf{x},t)}{c_{s0}}, \quad \mathbf{v}^* = \frac{\mathbf{v}}{U_\infty}, \quad Pe = \frac{U_\infty a}{D_e}. \tag{4}$$

Upon substitution into (3), the following dimensionless velocity field is obtained in the spherical coordinates:

$$v_r^* = -\left[1-\frac{1}{(r^*)^3}\right]\cos\theta, \quad v_\theta^* = \left[1+\frac{1}{2(r^*)^3}\right]\sin\theta \tag{5}$$

and the governing equation of the contaminant transport process becomes in dimensionless form:

$$\frac{\partial c^*}{\partial t^*} + Pe\,\mathbf{v}^*\cdot\nabla^* c^* = \nabla^{*2}c^* \tag{6}$$

with initial and boundary conditions defined as follows:

$$c^*(\mathbf{x}^*,t^*) = 0 \quad \text{at } |\mathbf{x}^*| > 1, \quad t^* = 0$$
$$c^*(\mathbf{x}^*,t^*) = H(t^*) \quad \text{at } |\mathbf{x}^*| = 1, \quad t^* > 0$$
$$c^*(\mathbf{x}^*,t^*) \to 0 \quad \text{as } |\mathbf{x}^*| \to \infty. \tag{7}$$

Equations (5)–(7) pose an unsteady convection-diffusion problem. It is expected that, at low Peclet numbers, the diffusion process will dominate in the vicinity of the sphere. However, as long as Pe is finite, at distances far from the sphere, there will always be a region, where the convection part of the process becomes significant and may even dominate the process. It is apparent that the problem has two length scales and hence, the regular perturbation technique cannot be applied to it. For this reason, we will use a singular perturbation method in the space and time domains. A similar method was introduced by Proudman and Pearson [8] for the steady-state motion of a sphere and a cylinder inside a viscous fluid. Bentwich and Miloh [9] and, later, Sano [10] extended the method to the problem of the unsteady momentum transfer of a sphere at low Reynolds numbers. We will adapt this method to the problem of unsteady mass transfer from a sphere in a saturated porous medium, and will obtain solutions for the transport of a scalar, the concentration of the contaminant and the instantaneous rate of mass transport. For this purpose, we decompose the time-space domain into four sub-domains: (i) short time and inner region (the immediate vicinity of the sphere); (ii) short time and outer region (far from the sphere); (iii) long time and inner region; and (iv) long time and outer region. The so-called 'Oseen distance', $L_I = a\,Pe^{-1}$, measured from the center of the sphere is the approximate boundary between the inner and outer regions.

Shortly after the inception of the leakage process, the contaminant diffusion and associated effects on the porous medium are confined to the inner region, while the outer region is undisturbed. At short times, the convective term of the governing equation may be neglected in the entire flow field and, hence, a classical perturbation method may be applied to this problem. The space–time sub-domains that were defined above as (i) and (ii) may be combined together as one, namely the short-time sub-domain. The solution in the short-time sub-domain is constructed by satisfying the boundary conditions at the surface of the sphere and at infinity. The development of the other two solutions in the sub-domains (iii) and (iv) is made according to the following matching conditions:

(a) the inner expansion $C^{(i)}$ satisfies the boundary condition on the surface of the sphere;

(b) the outer expansion $C^{(0)}$ satisfies the boundary condition at infinity;
(c) both inner and outer expansions match identically in the overlapping regions, where both expansions are expected to be valid;
(d) the long-time expansions match the short time expansions in the overlapping time domain.

For convenience, in the equations that follow, we will omit the superscript * of the dimensionless variables. It must be remembered, however, that the variables used hitherto are dimensionless.

3. Short-time expansion

Shortly after the inception of the process, the mass transfer from the sphere is dominated by the unsteady diffusion. The time variable in this sub-domain will be denoted by the lower case letter t. The concentration of the contaminant in the fluid may be given by a regular expansion of the concentration function $c(\mathbf{x}, t)$ as follows:

$$c = c_0 + Pe\, c_1 + \cdots. \tag{8}$$

It must be pointed out that in this case, the convection at the far field has not commenced and, therefore, all terms pertinent to this mode of heat transfer may be neglected. For this reason, a first order expansion of the concentration function is sufficient for the development of the solution. Substituting the last expression into equation (6), we obtain the governing equations for the first two order expansions of the concentration field as follows:

$$\nabla^2 c_0 = \frac{\partial c_0}{\partial t} \tag{9a}$$

$$\nabla^2 c_1 = \frac{\partial c_1}{\partial t} + \mathbf{v} \cdot \nabla c_0. \tag{9b}$$

A solution of the zeroth order expansion c_0, for the concentration field, may be obtained by using Laplace transforms. The solution of the equation, which satisfies the pertinent boundary conditions in the time domain, is:

$$c_0(r, t) = \frac{1}{r}\operatorname{erfc}\left(\frac{r-1}{2\sqrt{t}}\right) = \frac{1}{r}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{r-1}{2\sqrt{t}}} e^{-u^2}\, du\right] \tag{10}$$

where erfc denotes the complementary error function.

From (9b) and (10) we obtain the following expression for c_1 in the Laplace domain:

$$(\nabla^2 - s)\bar{c}_1 = \left(1 - \frac{1}{r^3}\right)\frac{1+\sqrt{sr}}{sr^2} e^{-\sqrt{s}(r-1)} P_1(\zeta) \tag{11}$$

where $P_n(\zeta)$ is the Legendre function of order n and $\zeta = \cos\theta$.

The last equation was solved by the method of 'variation of parameter'. After applying the boundary conditions for c_1 both at surface of the sphere and infinity, the solution for c_1 becomes as follows:

$$c_1(r, \zeta, t) = \left[\left(-\frac{1}{2} + \frac{3}{4r^2} - \frac{1}{4r^3}\right)\operatorname{erfc}\left(\frac{r-1}{2\sqrt{t}}\right) + \frac{3}{4}\left(\frac{1}{r} - \frac{1}{r^2}\right) e^{r-1+t}\operatorname{erfc}\left(\frac{r-1}{2\sqrt{t}} + \sqrt{t}\right)\right] P_1(\zeta). \tag{12}$$

A glance at (12) proves that the expression for the first-order contribution to the mass transfer is symmetric in ζ. Therefore, the net contribution of c_1 to the mass transfer coefficient (which results from a surface integral of the gradient of c_1) is zero.

4. Long-time, inner region expansion

Since the diffusion process dominates in the vicinity of the sphere, the convection in the inner region may be neglected in this sub-domain at all times. In this case, it is convenient to scale the dimensionless time in a different manner, which will be denoted by the capital letter T. The equation, which defines T, is as follows:

$$T = Pe^m\, t. \tag{13}$$

It will be proven in the next section that the parameter m must be equal to two. Hence, the following expression for the concentration field at long times and in the vicinity of the sphere, $C^{(i)}$, is obtained:

$$\nabla^2 C^{(i)} = Pe^2 \frac{\partial C^{(i)}}{\partial T} + Pe\, \mathbf{v} \cdot \nabla C^{(i)} \tag{14}$$

where the superscript (i) denotes that the function is pertinent to the inner region of the sphere. In this subdomain, we choose a second-order expansion for the concentration field. The concentration function is then represented by the following expression:

$$C^{(i)} = C_0^{(i)} + Pe\, C_1^{(i)} + Pe^2 C_2^{(i)} + \cdots. \tag{15}$$

Substituting (15) into (14) we deduce from the balance of the several powers of Pe the following three differential equations for the functions $C_n^{(i)}$ $(n = 0, 1, 2)$:

$$\nabla^2 C_0^{(i)} = 0, \quad \nabla^2 C_1^{(i)} = \mathbf{v} \cdot \nabla C_0^{(i)},$$
$$\nabla^2 C_2^{(i)} = \mathbf{v} \cdot \nabla C_1^{(i)} + \frac{\partial C_0^{(i)}}{\partial T} \tag{16a–c}$$

The (dimensionless) initial conditions at the surface of the sphere for the three functions $C_n^{(i)}$ $(n = 0, 1, 2)$ are: $C_0^{(i)} = H(T)$, where $H(T)$ is the Heavyside step function, $C_1^{(i)} = 0$ and $C_2^{(i)} = 0$. It must also be pointed out that, since a solution is sought for the inner region, only the boundary conditions at the surface of the sphere need to be satisfied, in the solution of the equations and that the boundary conditions at the far field need not be satisfied.

5. Long-time, outer region expansion

In this sub-domain the contaminant has already entered the outer region by diffusion. Now, both the diffusion and advection processes are of the same order of magnitude, even as $Pe \to 0$. Hence, we scale both the time variable and length variable as follows:

$$R = Pe^n r, \quad T = Pe^m t. \tag{17}$$

Substituting the scaled variables into the dimensionless mass transfer equation, we obtain the following expression for the concentration field in this sub-domain $C^{(o)}$:

$$Pe^{2n} \nabla_R^2 C^{(o)} = Pe^m \frac{\partial C^{(o)}}{\partial T} + Pe^{n+1} \mathbf{v} \cdot \nabla_R C^{(o)} \tag{18}$$

where the superscript (o) denotes that the function pertains to the outer region. It is obvious, that the only choice, which retains both the conduction and the advection terms, is $n = 1$ and $m = 2$. This result was anticipated in the previous section.

The concentration field $C^{(o)}$ in this sub-domain may again be represented by an asymptotic expansion with gauge functions $F_n^{(o)}(Pe)$. The gauge functions are unknown and will be determined by the matching requirements in the time-space domain:

$$C^{(o)} = F_0^{(o)}(Pe) C_0^{(o)} + F_1^{(o)}(Pe) C_1^{(o)} + \cdots. \tag{19}$$

Upon substitution of $C^{(o)}$ in the governing equation (18), we obtain the same differential equation for both the expansion functions $C_0^{(o)}$ and $C_1^{(o)}$ regardless of the gauge functions. The resulting expressions in the spherical coordinates may be written as follows:

$$\nabla_R^2 C_j^{(o)} = \frac{\partial C_j^{(o)}}{\partial T} - \zeta \frac{\partial C_j^{(o)}}{\partial R} - \frac{1-\zeta^2}{R} \frac{\partial C_j^{(o)}}{\partial \zeta}, \quad j = 0, 1. \tag{20}$$

In order to obtain the solution for equation (20) we transform it in the Laplace domain and use Goldstein's [11] transformation:

$$\overline{C_j^{(o)}}(R, \zeta) = \exp\left(\frac{-R\zeta}{2}\right) G(R, \zeta), \quad j = 0, 1 \tag{21}$$

to simplify the resulting expression. Hence, the differential equation for the function $G(R, \zeta)$ assumes a simpler form:

$$\nabla_R^2 G - (S + \tfrac{1}{4}) G = 0. \tag{22}$$

Since (22) is an equation for the outer region, it must satisfy the boundary conditions far from the sphere, but not necessarily the conditions on the surface of the sphere. A general solution to the last differential equation, which satisfies the boundary conditions at infinity, yields the following expression for the Laplace transforms of the functions $C_j^{(o)}$:

$$\overline{C_j^{(o)}} = \exp\left(\frac{-R\zeta}{2}\right) \sum_{n=0}^{\infty} H_n^j(S) [i_{-n-1}(\kappa R) - i_n(\kappa R)] P_n(\zeta), \quad j = 0, 1 \tag{23}$$

where $\kappa = (S + 1/4)^{1/2}$ and the functions $H_n^j(S)$ $(j = 0, 1)$ will be determined by the matching requirements. The symbols $i_n(x)$ denote the modified spherical Bessel functions, which are related to the ordinary spherical Bessel functions $I_n(x)$ by the expression:

$$i_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+1/2}(x), \quad n \text{ is integer}. \tag{24}$$

6. Solution of the equations and matching of the solutions

The matching procedure for the solutions is conceptually straightforward, but algebraically complicated. For this reason it will be presented in some detail. As matching conditions in the time domain, we simply require that the solutions in the short- and long-times match in their overlapping region. Essentially, this condition is a matching of the expression in equations (8) and (19) as t approaches large values and T tends to small values. The matching procedure may be performed in the Laplace domain by using the following property of the Laplace transforms:

$$\int_0^\infty \chi(t)\, e^{-st}\, dt = \int_0^\infty Pe^{-2} \chi(Pe^{-2} T)\, e^{-sT Pe^{-2}}\, dT = \int_0^\infty Pe^{-2} X(T)\, e^{-ST}\, dT \tag{25}$$

which implies,

$$s = Pe^2 S, \quad \bar{\chi}(s) = Pe^{-2} \bar{X}(S). \tag{26}$$

Hence, the matching requirement in the Laplace domain becomes:

$$Pe^2 [\overline{c_0} + Pe\, \overline{c_1} + \cdots]_{(s\to 0, r\to\infty, \zeta)} = [F_0(Pe) \overline{C_0^{(o)}} + F_1(Pe) \overline{C_1^{(o)}} + \cdots]_{(S\to\infty, R\to 0, \zeta)}. \tag{27}$$

Substituting the pertinent expressions in the above equation leads to the following conditions, which must be satisfied asymptotically:

$$F_0^{(o)}(Pe) = Pe, \quad Asmp_{S\ll 1} H_0^0(S) = \frac{1}{S}, \quad H_n^0(S) = 0 \quad \text{for } n \geqslant 1$$

$$F_1^{(o)}(Pe) = Pe^2, \quad Asmp_{S\ll 1} H_0^1(S) = \frac{1}{\sqrt{S}}, \quad H_n^1(S) = 0 \quad \text{for } n \geqslant 1. \tag{28}$$

It must be pointed out that the solution to the matching conditions for $H_0^0(S)$ and $H_0^1(S)$ are not unique. For example, $H_0^0(S) = 1/\sqrt{(S(S+\xi))}$, with ξ being an arbitrary constant, also satisfies the matching condition. The problem of non-uniqueness is resolved by ensuring that the resulting solution in the time domain satisfies the steady-state solution as $T \to \infty$. The steady solution in the outer region, $C^{(o)}_{\mathrm{Steady}}(R,\zeta)$ is as follows:

$$C^{(o)}_{\mathrm{Steady}}(R,\zeta) = \frac{Pe}{R}\exp\left[\frac{R}{2}(\zeta+1)\right] + \frac{Pe^2}{2R}\exp\left[\frac{R}{2}(\zeta+1)\right] + O(Pe^3). \quad (29)$$

The last equation imposes the following two requirements on the functions $H^0_0(S)$ and $H^1_0(s)$:

$$\lim_{S\to 0}[SH^0_0(S)] = 1 \quad \text{and} \quad \lim_{S\to 0}[SH^1_0(S)] = \tfrac{1}{2}. \quad (30)$$

Therefore, the only acceptable choices for $H^0_0(S)$ and $H^1_0(s)$ are:

$$H^0_0(s) = \frac{1}{S} \quad \text{and} \quad H^1_0(S) = \frac{\sqrt{S+\frac{1}{4}}}{S}. \quad (31)$$

Upon substitution in equation (23) and inversion in the time domain one obtains the following expressions for the functions $C^{(o)}_0$ and $C^{(o)}_1$:

$$C^{(o)}_0(R,\zeta,T) = \frac{e^{-\frac{R\zeta}{2}}}{R}\left[\frac{e^{-R/2}}{2}\mathrm{erfc}\left(\frac{R}{2\sqrt{T}} - \frac{\sqrt{T}}{2}\right) + \frac{e^{R/2}}{2}\mathrm{erfc}\left(\frac{R}{2\sqrt{T}} + \frac{\sqrt{T}}{2}\right)\right] \quad (32)$$

and

$$C^{(o)}_1(R,\zeta,T) = \frac{e^{-\frac{R\zeta}{2}}}{R}\left[\frac{e^{-\frac{1}{4}\left(T+\frac{R^2}{T}\right)}}{\sqrt{\pi T}} + \frac{e^{-R/2}}{4} \times \mathrm{erfc}\left(\frac{R}{2\sqrt{T}} - \frac{\sqrt{T}}{2}\right) - \frac{e^{R/2}}{4}\mathrm{erfc}\left(\frac{R}{2\sqrt{T}} + \frac{\sqrt{T}}{2}\right)\right]. \quad (33)$$

At this stage, we may perform the matching expansion in the inner sub-domain at long times from the inception of the process, with the functions $C^{(i)}_n$ ($n = 0, 1, 2$). From the governing equation for this sub-domain and the boundary conditions at the surface of the sphere we obtain:

$$C^{(o)}_0 = \left\{\frac{A_0(T)}{r} + [H(T) - A_0(T)]\right\}P_1(\zeta) + \sum_{n=1}^{\infty} A_n(T)\left(\frac{1}{r^{n+1}} - r^n\right)P_n(\zeta) \quad (34)$$

where $A_n(T)$ are functions of time, to be determined by the matching conditions. The matching requirement in this case yields the condition:

$$[C^{(i)}_0 + Pe\,C^{(i)}_1 + Pe^2\,C^{(i)}_2 + \cdots]_{(T,r\to\infty,\zeta)} = [F_0(Pe)C^{(0)}_0 + F_1(Pe)C^{(0)}_1]_{(T,R\to 0,\zeta)}. \quad (35)$$

The asymptotic limit of the right-hand side of (35) [to be denoted as RHS(35)] may be obtained in a straightforward manner:

$$\mathrm{RHS}(35) = Pe\left\langle \frac{H(T)}{R} + \left[W(T)\frac{H(T)}{2}P_1(\zeta)\right] + R\left[\frac{H(T)}{6} + \frac{W(T)}{2}P_1(\zeta) + \frac{H(T)}{12}P_2(\zeta)\right]\right\rangle + Pe^2\left\langle\frac{W(T)}{R} + \left[\frac{H(T)}{4}\frac{W(T)}{2}P_1(\zeta)\right]\right\rangle + O(Pe^3) \quad (36)$$

where the function $W(T)$ is defined as follows:

$$W(T) = \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{T}}{2}\right) + \frac{e^{-\frac{T}{4}}}{\sqrt{\pi T}} = \frac{1}{\sqrt{\pi}}\int_0^{\frac{\sqrt{T}}{2}} e^{-u^2}\,du + \frac{e^{-\frac{T}{4}}}{\sqrt{\pi T}}. \quad (37)$$

We replace $r = R/Pe$ in equation (34) and then substitute the result in the left-hand side of (35), which is denoted as LHS(35). Hence, we obtain the following expression:

$$\mathrm{LHS}(35) = \langle H(T)A_0(T)\rangle + Pe\left\langle\frac{A_0(T)}{R}\right\rangle + \sum_{n=1}^{\infty} A_n(T)\left(\frac{Pe^{n+1}}{R^{n+1}}\frac{R^n}{Pe^n}\right)P_n(\zeta) + O(Pe^2). \quad (38)$$

By matching the terms of (38) with the terms of the corresponding orders of Pe in equation (36), we determine the unknown functions $A_n(T)$:

$$A_0(T) = H(T), \quad A_n(T) = 0 \quad \text{for } n \geqslant 1. \quad (39)$$

Therefore, the final expression for the zeroth-order solution in the inner region becomes:

$$C^{(i)}_0(T,r,\zeta) = \frac{H(T)}{r}P_1(\zeta). \quad (40)$$

Now, from the governing equation for $C^{(i)}_1$ [equation (16b)], we obtain the following differential equation for this function:

$$\nabla^2 C^{(i)}_1 = \left(\frac{H(T)}{r^2} - \frac{H(T)}{r^5}\right)P_1(\zeta). \quad (41)$$

The general solution of (40) that satisfies the boundary conditions at the surface of the sphere is:

$$C^{(i)}_1 = B_0(T)\left(\frac{1}{r} - 1\right) + \left[-\frac{H(T)}{2} - \frac{H(T)}{4r^3} + \frac{B^1(T)}{r^2} - \left(B_1(T) - \frac{3H(T)}{4}\right)r\right]P_1(\zeta) + \sum_{n=0}^{\infty} B_n(T)\left(\frac{1}{r^{n+1}} - r^n\right)P_n(\zeta) \quad (42)$$

where the coefficients $B_n(T)$ may also be deduced from the matching requirements. The results of the matching procedure are:

$$B_0(T) = W(T), \quad B_1(T) = \frac{3H(T)}{4} \quad \text{and} \quad B_n(T) = 0, \quad \text{for } n \geqslant 2. \tag{43}$$

Hence, the functional form of $C_1^{(i)}$ becomes:

$$C_1^{(i)} = \left(\frac{1}{r}-1\right)W(T) - \left(\frac{1}{2}+\frac{1}{4r^3}-\frac{3}{4r^2}\right)H(T)P_1(\zeta). \tag{44}$$

From the expressions of $C_0^{(i)}$ and $C_1^{(i)}$ and the governing equation (16c), one may derive a differential equation for $C_2^{(i)}$:

$$\nabla^2 C_2^{(i)} = \mathbf{v}\cdot\nabla C_1^{(i)} + \frac{\partial C_0^{(i)}}{\partial T} \equiv \sum_{n=0}^{2} Z_n(r,T)P_n(\zeta) \tag{45}$$

where the functions $Z_j(r, T)$ are as follows,

$$Z_0(r,T) = \left[\frac{1}{3r}+\frac{1}{12r^4}-\frac{3}{4r^6}+\frac{1}{3r^7}\right]H(T)$$

$$Z_1(r,T) = \left(\frac{1}{r^2}-\frac{1}{r^5}\right)W(T)$$

$$Z_2(r,T) = \left[\frac{1}{3r}+\frac{3}{2r^3}-\frac{5}{6r^4}-\frac{3}{4r^6}+\frac{5}{12r^7}\right]H(T). \tag{45a}$$

The general form of the solution of (45), which is subject to the boundary conditions at the surface of the sphere, is:

$$C_2^{(i)} = \sum_{n=0}^{2}\left[\frac{E_n(T)-L_n(1,T)}{r^{n+1}} - E_n(T)r^n + L_n(r,T)\right]P_n(\zeta) + \sum_{n=3}^{\infty} E_n(T)\left(\frac{1}{r^{n+1}}-r^n\right)P_n(\zeta) \tag{46}$$

where the functions $L_n(r, T)$ are given by the following expressions:

$$L_0(r,T) = \left[\frac{r}{6}+\frac{1}{24r^2}-\frac{1}{16r^4}+\frac{1}{60r^5}\right]H(T),$$

$$L_1(r,T) = \left(\frac{1}{2}+\frac{1}{4r^3}\right)W(T),$$

$$L_2(r,T) = \left[\frac{r}{12}-\frac{1}{4r}+\frac{5}{24r^2}-\frac{1}{8r^4}+\frac{5}{168r^5}\right]H(T). \tag{47}$$

As with the previous functions of T, the functions $E_n(T)$ may be determined by the matching conditions, which yield the following results:

$$E_0 = \frac{H(T)}{4}, \quad E_1 = \frac{1}{2}W(T), \quad E_n = 0 \quad \text{for } n \geqslant 2. \tag{48}$$

Hence, the final expression for $C_2^{(i)}$ in the time domain becomes:

$$C_2^{(i)} = \left(\frac{r}{6}-\frac{1}{4}+\frac{7}{80r}+\frac{5}{24r^2}-\frac{1}{16r^4}+\frac{1}{60r^5}\right)H(T) + \left(\frac{r}{2}-\frac{1}{2}+\frac{1}{4r^2}-\frac{1}{4r^3}\right)W(T)P_1(\zeta) + \left(\frac{r}{12}-\frac{1}{4r}+\frac{5}{24r^2}+\frac{3}{56r^3}-\frac{1}{8r^4}+\frac{5}{168r^5}\right) \times H(T)P_2(\zeta). \tag{49}$$

7. The concentration field and rate of mass transfer

The practical applications of the problem at hand (e.g., for risk assessment or cleanup processes) require information on the spread of the contaminant in the porous medium and of the rate of mass transfer. Hence, the main objective for the solution in this manuscript is the determination of the contaminant concentration in the porous medium and the instantaneous rate of mass transport from the sphere. Summarizing the results of the previous sections, one may now obtain solutions for the concentration field in the four sub-domains of the problem.

First, during the short time sub-domain [$t = O(1)$] for both inner and outer region, where the diffusion process dominates, the unsteady concentration field, correct to $O(Pe^2)$, is:

$$c(r,\zeta,t) = \frac{1}{r}\operatorname{erfc}\left(\frac{r-1}{2\sqrt{t}}\right) + Pe\left\langle\left[\left(-\frac{1}{2}+\frac{3}{4r^2}-\frac{1}{4r^3}\right)\operatorname{erfc}\left(\frac{r-1}{2\sqrt{t}}\right) + \frac{3}{4}\left(\frac{1}{r}-\frac{1}{r^2}\right)\exp(r-1+t)\operatorname{erfc}\left(\frac{r-1+\sqrt{t}}{2\sqrt{t}}\right)\right]P_1(\zeta)\right\rangle + O(Pe^2), \quad t = O(1). \tag{50}$$

Second, at long times from the inception of the process, and for the inner-region, where $t = O(Pe^{-2})$ and $r = O(1)$, the concentration function $C^{(i)}$ is given by the combination of the expressions for $C_0^{(i)}$, $C_1^{(i)}$ and $C_2^{(i)}$, which yield the following expression:

$$C^{(i)} = \frac{H(t)}{r} + Pe\left\langle\left[\frac{\operatorname{erf}\left(\frac{Pe\sqrt{t}}{2}\right)}{2} + \frac{e^{-\frac{Pe^2}{4}}}{Pe\sqrt{\pi t}}\right]\left(\frac{1}{r}-1\right) - \left(\frac{1}{2} - \frac{3}{4r^2} + \frac{1}{4r^3}\right)H(t)\cos\theta\right\rangle$$

$$+ Pe^2\left(\frac{r}{6} - \frac{1}{4} + \frac{7}{80r} + \frac{5}{24r^2} - \frac{1}{16r^4} + \frac{1}{60r^5}\right)H(t)$$

$$+ \left(\frac{r}{2} - \frac{1}{2} + \frac{1}{4r^2} - \frac{1}{4r^3}\right)\left[\frac{\operatorname{erf}\left(\frac{Pe\sqrt{t}}{2}\right)}{2} + \frac{e^{-\frac{Pe^2 t}{4}}}{Pe\sqrt{\pi t}}\right]\cos\theta$$

$$+ \left[\left(\frac{r}{12} - \frac{1}{4r} + \frac{5}{24r^2} + \frac{3}{56r^3} - \frac{1}{8r^4} + \frac{5}{168r^5}\right) \times H(t)\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\right] + O(Pe^3). \quad (51)$$

Third, for the sub-domain of long times and outer region, where $t = O(Pe^{-2})$ and $r = (Pe^{-1})$ the following expression is derived:

$$C^{(o)} = Pe\left\langle\frac{1}{Pe\,r}\left[\frac{e^{-\frac{Pe\,r}{2}(1+\cos\theta)}}{2}\operatorname{erfc}\left(\frac{r}{2\sqrt{t}} - \frac{Pe\sqrt{t}}{2}\right) + \frac{e^{\frac{Pe\,r}{2}(1-\cos\theta)}}{2}\operatorname{erfc}\left(\frac{r}{2\sqrt{t}} + \frac{Pe\sqrt{t}}{2}\right)\right]\right\rangle$$

$$+ Pe^2\left\langle\frac{1}{Pe\,r}\left[\frac{e^{-\frac{Pe\,r}{2}\cos\theta - \frac{1}{4}\left(Pe^2 t + \frac{r^2}{t}\right)}}{Pe\sqrt{\pi t}} + \frac{e^{-\frac{Pe\,r}{2}(1+\cos\theta)}}{4}\operatorname{erfc}\left(\frac{r}{2\sqrt{t}} - \frac{Pe\sqrt{t}}{2}\right) - \frac{e^{\frac{Pe\,r}{2}(1-\cos\theta)}}{4}\operatorname{erfc}\left(\frac{r}{2\sqrt{t}} + \frac{Pe\sqrt{t}}{2}\right)\right]\right\rangle + O(Pe^3). \quad (52)$$

The dimensionless Sherwood number (or its equivalent, the Nusselt number in the problem of heat transfer) is a commonly used parameter in engineering applications for the determination of the rate of mass transfer. The Sherwood number is defined in terms of the dimensionless variables used so far in this study, as follows:

$$Sh = -\frac{\int_A \nabla c \cdot \mathbf{n}\, dA}{2\pi}. \quad (53a)$$

Therefore, from the last three equations, we may obtain expressions for the time-dependent Sherwood number in all the sub-domains of the problem. From equation (49) the unsteady Sherwood number at short times from the inception of the process is calculated as follows:

$$Sh = 2\left(1 + \frac{1}{\sqrt{\pi t}}\right) + O(Pe^2), \quad t = O(1). \quad (53b)$$

It must be emphasized that the degree of accuracy of equation (53b) is of the order Pe^2, as indicated. This occurs, because the contaminant concentration expansions in c_0 and c_1, from which the last expression is obtained, are correct to the order of Pe^1. However, the c_1 term, which is proportional to Pe, does not contribute at all to the total rate of mass transfer, because it is symmetric with respect to the surface of the sphere.

The Sherwood (or Nusselt) number expressions in this case exhibits the typical $(\pi t)^{-1/2}$ behavior of the diffusion processes. Thus, expression (53b) is almost the same as the expressions derived by Taylor and Acrivos [12] and Feng and Michaelides [13, 14] in the case of heat transfer from a sphere in an infinite fluid. The only difference between the two cases, is that equation (53b) is correct to $O(Pe^2)$, while that for the heat transfer is valid to $O(Pe^{1+})$. Given that in this sub-domain the molecular diffusion dominates entirely the processes of heat and mass transport, it is of no surprise that the last expression is almost the same as the expression of the Nusselt number in the analogous problem of unsteady heat transfer. The fact that the two agree, is a validation that the expression derived for the concentration field at short times is correct.

Similarly, one may calculate the Sherwood number at long times from the inception of the process, which is:

$$Sh = 2\left\{1 + Pe\left[\frac{1}{2}\operatorname{erf}\left(Pe\frac{\sqrt{t}}{2}\right) + \frac{1}{Pe\sqrt{\pi t}}\exp\left(-\frac{Pe^2\,t}{4}\right)\right] - \frac{13}{80}Pe^2\right\} + O(Pe^3), \quad t = O(Pe^{-2}). \quad (54a)$$

In the case of long times the resulting expression is correct to the order Pe^3 as indicated. When compared to the analogous problem of heat transfer from a particle at low Reynolds numbers [13, 14], one observes that the first three terms of equation (54) are similar to the analogous terms for the Nusselt number. The main difference is in the last term, which is entirely due to the type of the velocity field obtained, which affects the advection part of the process. This agreement in the two processes leads us to believe that the concentration fields derived for the long-time sub-domains are accurate. In addition, the steady-state solution for the Sherwood number (which is obtained as $t \to \infty$) is:

$$Sh = 2 + 3Pe - 13/40Pe^2. \quad (54b)$$

This expression agrees exactly with the well-known solution, obtained under the creeping flow condition

($Pe = 0$), which is $Sh_{(Pe=0)} = 2$. Thus, equation (54b) may be thought of as the extension of the classical creeping flow solution in the case of finite Peclet numbers.

It is observed that both the solutions for short- and long-times include a function of time, which is essentially a history term describing the contaminant transport process, from its inception to the current state. In the pure diffusion problem (short times) this term exhibits the typical time dependence of diffusion processes and varies at $t^{-1/2}$. The same mathematical behavior has been observed in the unsteady momentum and heat transfer processes from a sphere to a fluid [12–17] using the creeping flow hypothesis ($Re \to 0$ or $Pe \to 0$) where the diffusion processes dominate in the whole time and space domain. In the case of finite Re or Pe, where the advection processes become significant, at long times from the inception of the process, the history terms become more complex. The effect of the advection process is to accelerate the dissipation of these history terms [14–17]. As seen in the last equation, the history terms dissipate at the faster rate of $t^{-1/2}\exp(-t)$ in the case of contaminant diffusion/advection. An interesting result in this latter case, as shown in equation (54a), is that the term of the order Pe^2 is time-independent, despite the fact that the concentration distribution depends on time.

Numerical calculations were made for the concentration fields, given by equations (50)–(52). In all cases it was assumed that $a = 1$, and $Pe = 0.25$, which results in $L_I = 4$ (the measure of the overlapping area between the inner and outer regions). The initial concentration condition is a unit step change at the surface of the sphere. This corresponds to the sudden commencement of the leakage at $t = 0$. Figure 2 shows the concentration field, which results in the inner region for short times. The left-hand side of the graph is for dimensionless time $t = 1$ and the right-hand side for $t = 2$ (times are made dimensionless by dividing with the characteristic time of the diffusion process). The calculations show the level of contaminant concentration. It is observed that very close to the surface also of the sphere ($r < 1.3$) the concentration lines do not vary significantly during this time interval. It is observed that, there is an appreciable change in the concentration lines far from the surface ($r > 2$). This is an indication that an almost steady-state condition is quickly established in the immediate vicinity of the sphere. However, the contaminant migrates at a faster rate towards the outer region.

Figure 3 depicts the inner region at dimensionless times $t = 20$ (left-hand side) and $t = 50$ (right-hand side). It is obvious that the combination of the diffusion and advection processes have brought the contaminant to the outer region. Significant time gradients are observed close to the boundary of the inner and outer regions ($2 < r < 4$). The asymmetric (top to bottom) aspects of the figure underline the relative significance of the advection, which is induced by the vertical flow of the outside fluid.

Figure 4 depicts only the outer region (the concentration field in the inner region is not shown) at long times from the inception of the process. The left-hand side is for $t = 20$ and the right-hand side for $t = 25$. The time derivatives are very significant in this sub-domain and indicate that the process of the spreading of the contaminant is progressing swiftly. The importance of the advection part of this process becomes apparent when

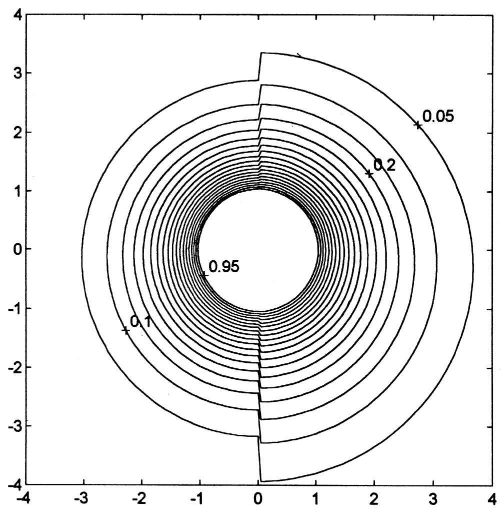

Fig. 2. Concentration distribution for the short-time, inner region sub-domain for $Pe = 0.25$. On the left side are results for dimensionless time $t = 1$ and on the right for $t = 2$.

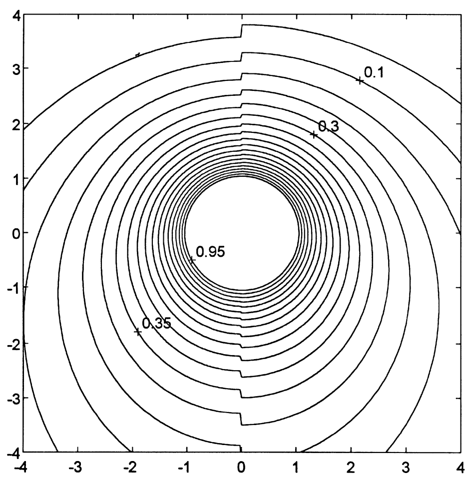

Fig. 3. Concentration distribution for long-time, inner region sub-domain for $Pe = 0.25$. On the left side are results for dimensionless time $t = 20$ and on the right for $t = 50$.

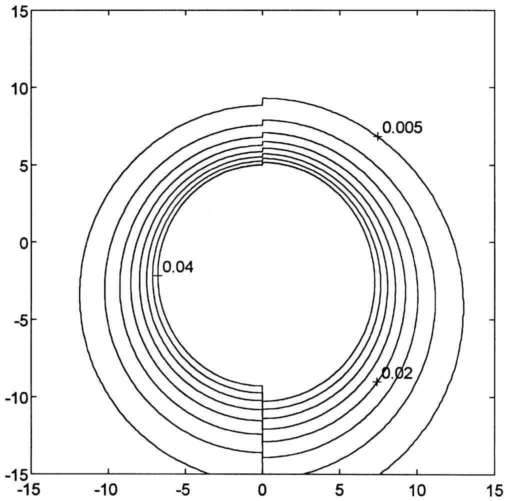

Fig. 4. Concentration distribution for the long-time, outer region sub-domain for $Pe = 0.25$. On the left side are results for dimensionless time $t = 20$ and on the right for $t = 25$.

we observe that the gradients are sharper and the contaminant is spreading faster in the bottom part of the figure than in the upper part.

8. Solutions for the case of constant mass flux at the surface

One may use the same technique that was described in the previous sections and obtain the solution to the unsteady convection problem from a sphere in a saturated porous medium, when there is a constant mass flux at the surface. In this case, the mass flux is normalized, by the condition $(-\nabla c \cdot n)_{r=1} = 1$ at the surface of the sphere and the same method is used to solve the problem under the constant mass flux condition. In the following paragraphs we will list the results of the calculations on the concentration distribution in the porous medium.

In the short-times domain, $[t = O(1)]$, the contaminant distribution is:

$$c(r,\zeta,t) = \frac{1}{r}G_0(r,t) + Pe\left\langle \left(\frac{1}{2}+\frac{1}{4r^3}\right)G_0(r,t) + \frac{3}{8r}G_1(r,t) + \frac{3}{8r^2}G_2(r,t)\right\rangle P_1(\zeta) + O(Pe^2), \quad t = O(1) \tag{55}$$

where the functions G_0, G_1 and G_2 are:

$$G_0(r,t) = Ert(0) - Ert(1),$$
$$G_1(r,t) = Ert(0) - \mathrm{Im}\left[\frac{1}{\Lambda_1}Ert(\Lambda_1) - \frac{1}{\Lambda_2}Ert(\Lambda_2)\right]$$
$$G_2(r,t) = \mathrm{Im}\,[Ert(\Lambda_1) - Ert(\Lambda_2)] \tag{56}$$

with

$$Ert(v) \equiv e^{v(r-1)+v^2 t}\,\mathrm{erfc}\left(\frac{r-1}{2\sqrt{t}} + v\sqrt{t}\right),$$

and $\Lambda_1 = \sqrt{2}\,e^{-j\frac{\pi}{4}}$, $\Lambda_2 = \sqrt{2}\,e^{j\frac{\pi}{4}}$, $j = \sqrt{-1}$.

At long times from the inception of the process $[t = O(Pe^{-2})]$ the distribution of the contaminant in the inner region, $[r = O(1)]$, is:

$$C^{(i)} = \frac{H(t)}{r} + Pe\left\langle -\left[\frac{1}{2}\mathrm{erf}\left(\frac{Pe\sqrt{t}}{2}\right) + \frac{e^{-\frac{Pe^2}{4}}}{Pe\sqrt{\pi t}}\right] - \left(\frac{1}{2} - \frac{3}{8r^2} + \frac{1}{4r^3}\right)H(t)\cos\theta\right\rangle$$
$$+ Pe^2\left\langle \left(\frac{r}{6} + \frac{1}{8r} + \frac{5}{24r^2} - \frac{1}{32r^4} + \frac{1}{60r^5}\right)H(t) + \left(\frac{r}{2} + \frac{1}{4r^2}\right)\left[\frac{1}{2}\mathrm{erf}\left(\frac{Pe\sqrt{t}}{2}\right) + \frac{e^{-\frac{Pe^2}{4}}}{Pe\sqrt{\pi t}}\right]\cos\theta\right.$$
$$\left. + \left(\frac{r}{12} - \frac{1}{8r} + \frac{5}{24r^2} - \frac{1}{28r^3} - \frac{1}{16r^4} + \frac{5}{168r^5}\right) \times H(t)\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\right]\right\rangle + O(Pe^3). \tag{57}$$

Finally, the long-time, outer region $[t = O(Pe^{-2})$ and $r = O(Pe^{-1})]$ solution for the concentration function, becomes:

$$C^{(o)} = Pe\left\langle \frac{1}{Pe\,r}\left[\frac{e^{-\frac{Pe\,r}{2}(1+\cos\theta)}}{2}\mathrm{erfc}\left(\frac{r}{2\sqrt{t}} - \frac{Pe\sqrt{t}}{2}\right) + \frac{e^{\frac{Pe\,r}{2}(1-\cos\theta)}}{2}\mathrm{erfc}\left(\frac{r}{2\sqrt{t}} + \frac{Pe\sqrt{t}}{2}\right)\right]\right\rangle + O(Pe^3). \tag{58}$$

It must also be pointed out that in the case of constant mass flux and in the sub-domain of long times and outer region, the contribution of the second-order term $[O(Pe^2)]$ is zero. This indicates that the point-source solution is valid to the second order approximation in the outer region. Obviously, this is not the case with equation (52) that pertains to the condition of constant surface-concentration.

9. Conclusions

The problem of unsteady forced convection mass transport from a sphere, immersed in an unbounded fluid-saturated porous medium has two time scales and cannot be solved by a regular perturbation analysis. A singular perturbation method may be used to solve this problem. The resulting governing equations in four sub-domains are solved and general expressions for the unsteady concentration fields are obtained. Matching conditions at the boundaries of the sub-domains determine the coefficients of the solutions. The analytical expressions of the mass transfer show that the Sherwood number contains history-dependent terms, which are characteristics of diffusion/advection problems. Similar terms have been observed in the analogous cases of heat and momentum transfer from a sphere to an infinite fluid. The results show that a very thin boundary layer is established close to the surface of the sphere, where the velocity gradient is very sharp. Concentration gradients are established very fast in the immediate vicinity of the sphere and spread swiftly in the rest of the inner region. At long times the advection part of the process becomes significant, especially in the outer region of the flow.

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Appendix A: the velocity field around the sphere

From the governing equations of the problem we obtain:

$$\Psi = \frac{K}{\mu}(p+\rho g z), \quad \mathbf{v} = \nabla\Psi. \tag{A1}$$

Hence, Ψ satisfies the Laplace equation. Therefore, the velocity field is a solution to the potential flow problem. After using the boundary condition at infinity, the velocity field may be written as:

$$\mathbf{v} = -\mathbf{U} + \sum_{n=1}^{\infty} B_n(\cdot)^n \nabla^{n+1}\frac{1}{r} \tag{A2}$$

where $\mathbf{B}_n$ is an nth-order tensor, and $(\cdot)^n$ denotes n times the inner product. Utilizing the boundary condition at the surface of the sphere, and the condition

$$\nabla^2\frac{1}{r} = \frac{3}{r^3}\mathbf{nn} - \frac{1}{r^2}I \tag{A3}$$

we obtain $\mathbf{B}_1 = \mathbf{U}/2$, and $\mathbf{B}_n = \mathbf{0} (n > 1)$. These results yield the velocity field in (3).

Appendix B: tangential slip on the sphere

The Brinkman model takes into account the no slip condition on the surface of the sphere. Accordingly, the velocity field in the porous medium is described by the following expression:

$$\mathbf{v}_m = \frac{k}{\mu}\nabla(p+\rho g z) + k\nabla^2\mathbf{v}_m. \tag{B1}$$

This expression satisfies the no-slip conditions at the surface of the sphere:

$$\mathbf{v}_m \cdot \mathbf{n} = 0, \quad \mathbf{v}_m \cdot (I - \mathbf{nn}) = \mathbf{0} \quad \text{at } r = a; \quad \mathbf{v}_m = -\mathbf{U} \quad \text{at } r \to \infty. \tag{B2}$$

It has been shown [6] that an analytical expression of the Brinkman velocity field is:

$$\mathbf{v}_m = \left[1 - \left(1 + \frac{3}{\beta} + \frac{3}{\beta^2}\right)\frac{a^3}{r^3} + \left(\beta\frac{a^2}{r^2} + \frac{a^3}{r^3}\right)\frac{\mathrm{e}^{-\beta\left(\frac{r}{a}-1\right)}}{\beta^2}\right]\mathbf{nn}\cdot\mathbf{U}$$
$$+ \left[1 + \left(1 + \frac{3}{\beta} + \frac{3}{\beta^2}\right)\frac{a^{32}}{2r^3} - \left(\frac{a}{r} + \beta\frac{a^2}{r^2} + \frac{a^3}{r^3}\right)\frac{3\,\mathrm{e}^{-\beta\left(\frac{r}{a}-1\right)}}{2\beta^2}\right]$$
$$\times (I - \mathbf{nn})\cdot\mathbf{U} \tag{B3}$$

where $\beta = a^2/k$.

The exponential functions in (B3) denote the presence of a boundary layer near the surface of the sphere. A measure of the thickness of this boundary layer is a/β. In the cases we consider here, the radius of the sphere is $O(1\ \text{m})$, while the soil permeability k is $O(10^{-7}\ \text{m}^2)$. Therefore, $\beta = O(10^7)$ and the thickness of the boundary layer is less than a millionth of the sphere's radius. This size is also by far smaller than the inner region, as defined in Section 2. Inside the boundary layer, there is a very high velocity gradient in the tangential direction, with the velocity increasing rapidly from zero to the value calculated by Darcy's law. Because of the very small relative size of the boundary layer, it is acceptable to neglect it and to consider that there is a step velocity change from zero (no slip condition) to Darcy's slip velocity in the tangential direction to the surface.

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